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THE PRODUCTION OF TURBULENCE

By W. Tollmien

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It is known that for geometrically similar boundary surfaces the form of flow of liquids and gases, in so far as the effect of compressibility can be disregarded, is determined only by the Reynolds Number, that nondimensional or absolute quantity which is composed of a characteristic velocity and distance and the kinematic viscosity. The most conspicuous and important change in the form of flow with changing Reynolds Number is the sharp transition from the laminar to the turbulent state of flow. The Reynolds Number for this transition has been determined from numerous observations in many cases; for example, in the case of the flow through tubes or about a sphere. Still the summary announcement of this "critical" Reynolds Number is physically incomplete since it depends largely on the disturbances of the original laminar flow. From the disturbances we now have, at the worst, the statement of their point or origin, e.g., in the case of the flow through tubes, whether the disturbances proceed from the inflow or farther downstream from the walls, where they apparently have less effect than in the former case. Beyond this point, however, the disturbances have not yet been experimentally defined and classified. Hence the experiments leave it

*"Ueber die Entstehung der Turbulenz." From Nachrichten der Gesellschaft der Wissenschaften zu Göttingen (Report I of the Göttingen Scientific Society), Mathematisch-Physikalische Klasse, 1929, pp. 21-44.

uncertain in what sense one can speak accurately of a critical Reynolds Number as an upper limit for the laminar flow, and whether this flow can be maintained up to a large Reynolds Number, by continually reducing the magnitude of the disturbances.

We might hope to obtain an accurate insight into the production of turbulence by a photographic investigation of the changes in the flow. Reynolds' method has been much used, according to which the flow in glass tubes is observed with the aid of a colored thread suspended in the middle of the tube. Prandtl sought to obtain a clearer idea of the process in the smooth flow of water in an open channel by photographing aluminum powder strewn on the surface.* By this means he succeeded in determining approximately the location of the flow transformation, e.g., in the flow through a tube, mostly in the so-called starting region, the results in this and other cases being similar to those obtained by pressure and velocity measurements. Moreover, from the complicated pictures, it was possible to give a description only in rough outlines. Hitherto the production of turbulence could not be further elucidated experimentally.

In any case, according to the customary view, the production of turbulence begins with the laminar flow becoming unstable. The simplest way to obtain a theoretical conception of this process is to assume a level laminar velocity distribution (which, moreover, can depend only on the coordinate perpendicu-

*L. Prandtl, Z. f. angew. Math. u. Mech., Vol. I (1921), p. 431, and Phys. Z., Vol. 23 (1922), p. 19.

lar to the direction of flow) and then to subject this velocity profile to two-dimensional disturbances. The first profile to be treated in this way by the method of small oscillations was that of the "Couette flow" with linear dependence on the velocity of the transverse coordinate. The result, obtained by Hopf* and Mises**, gave no indication of the production of turbulence, but a disappearance of all disturbances, and consequent permanent stability of the assumed laminar flow. The simplification of the disturbances was first obtained by Prandtl (See footnote, page 2) and Tietjens*** for certain profiles. These investigators, however, obtained the physically surprising result that, within the range of the Reynolds Numbers for which the approximations applied, it always yielded simplified oscillations, while it was to be expected that, below a certain Reynolds Number, the laminar flow would be stable despite all disturbances. In the Prandtl-Tietjens investigations, for mathematical simplicity according to a suggestion of Rayleigh, profiles were assumed for the fundamental flow, which were composed of straight sections. The universal disappearance of the curvature of the profile of the fundamental flow indicates, however, in this case, such an essential physical assumption that, by allowing for the curvature, a modification of the Prandtl-Tietjens result might be expected at the physically still unsatisfactory points. In

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*L. Hopf, Ann. d. Phys., Vol. 44 (1914), p. 1.
**R. v. Mises, "Beitrag zum Oscillations problem," Heinrich Weber Festschrift, 1912, p. 252.
***O. Tietjens, Z. f. angew. Math. u. Mech., Vol. V (1925), p. 200.

what follows, therefore, profiles are used which do not have a universally disappearing curvature.

This introduction is, of course, not in the least intended to give a comprehensive survey of the extensive literature* on the production of turbulence. It is only intended to indicate the connection in which the present work was undertaken. I am greatly indebted to Dr. Prandtl for suggesting this investigation and for his encouragement and interest.

I. General Disturbance Equation

The principal flow is represented by $U(y)$, y being the coordinate perpendicular to the direction of flow in the x axis. The disturbances are resolved according to Fourier, so that the flow function of a given Fourier term $\varphi(y)\, e^{i(\alpha x - \beta t)} = \varphi(y)\, e^{i\alpha(x - ct)}$. Then α represents the spatial cycle, the real part of β represents the periodic cycle, and the real part of c represents the phasic velocity of the given disturbance. The imaginary part of β indicates the increasing or damping of the disturbance according to whether it is positive or negative. According to the method of small vibrations, only the linear terms in $\varphi(y)$ are always considered. Then, by introduction into the Navier-Stokes differential equations for viscous fluids, elimination of the pressure and general introduction of absolute quantities, we obtain the disturbance equation

$$(U - c)(\varphi'' - \alpha^2 \varphi) - U'' \varphi = -\frac{i}{\alpha R}(\varphi'''' - 2\alpha^2 \varphi'' + \alpha^4 \varphi) \qquad (1)$$

*See footnote, page 5.

in which R is the Reynolds Number. The reference length, with which a characteristic velocity of the profile (e.g., the maximum U_m) multiplied after division by the kinematic viscosity yields νR, is so selected as a dimension of the profile, that the deductions of the absolute U in equation (1) are of the order of magnitude of 1. The disappearance of both components (φ and φ') of the disturbance is a marginal condition at the boundary walls, i.e., the limits of $U(y)$ by definition.

At first we will assume β and c to be real, and will seek the points of transition from stability to instability. For this problem two statements can now be made concerning c and α. If we write the disturbance equation symbolically $L(\varphi) = 0$ and integrate the expressions $\overline{\varphi}\, L(\varphi) - \varphi\, \overline{L(\varphi)}$ and $\overline{\varphi}\, L(\varphi) + \varphi\, \overline{L(\varphi)}$ (the conjugated complex quantities being overlined) between the boundaries with allowance for the boundary conditions, we then obtain, in the simple and well-known manner, $c < U_{max}$ for $U'' \leqq 0$ and also $\alpha << R$.** Just the circumstance that we must accordingly assume a zero position of $U - c$, necessitates special mathematical expressions and is also accompanied by a physically interesting behavior of the disturbances.

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*Footnote from page 4:
F. Noether, Z. f. angew. Math. u. Mech. 1, page 125, 1921;
L. Schiller, Phys. Z. 26, page 566, 1925.

**Solberg, "The Turbulence Problem," Proc. of the First International Congress for Applied Mechanics, Delft, 1924, page 387.
Noether, "Zur asymptotischen Behandlung der stationären Lösungen im Turbulenzproblem," Z. f. angew. Math. u. Mech., Vol. 6, 1926, page 232.

II. Frictionless Solutions of the Disturbance Equation

In general αR is assumed to be very large. It is natural to endeavor to represent the solutions of the disturbance equation in such manner that the terms multiplied by the very small factor $\frac{1}{\alpha R}$ can be omitted, so that they are equivalent to solutions of the frictionless disturbance equation.

$$(U - c)\ (\varphi'' - \alpha^2\ \varphi) - U''\ \varphi = 0$$

We will consider the solutions of this frictionless disturbance equation without at first going further into their connection with the solutions of the general disturbance equation (for large αR).

At the position $U = c$, the differential equation (2) has a singular point. Since $\frac{U''}{U - c}$ has a pole of the first order in this position, provided U'' does not vanish at just this point, we have a nonessential singularity (point of precision) under consideration, and we can easily establish, according to general theorems regarding linear differential equations, the converging series of integrals from the point $U - c = 0$. At this point let $y = 0$ and y be positive for $U - c > 0$. Accordingly the values at this point are indicated by the subindex C. There are then two linear independent solutions of equation (2), when the potential series in y are designated by $P_1(y)$ and $P_2(y)$, and when the constant determined by the differential equation is designated by A.

$$\varphi_1 = y\, P_1(y) = y + \ldots$$

$$\varphi_2 = P_2(y) + A\,\varphi_1 \log y = 1 + \ldots + \frac{U''_0}{U'_0}\, y \log y + \ldots$$

This establishes the fact once for all that $\log y$ is real for positive y.

Before drawing conclusions from the analytical character of φ_1 and φ_2, these should be determined for a special case of U and indeed for a parabolic velocity distribution. Accordingly, $U - c = y\,(2a - y)$, so that $U' = 0$ for $y = a$. Then $U = 1$, if we put $a = \sqrt{1 - c}$, and $U = 0$ for $y = a - 1$. We expediently introduce $y_1 = y/a$ and $\alpha_1 = a\,\alpha$, so that $y_1 = 1$ at the vertex of the parabola, and we then obtain

$$y_1\,(2 - y_1)\,(\varphi'' - \alpha_1^2\,\varphi) + 2\,\varphi = 0$$

as the differential equation in the new variables. We then obtain the series developments

$$\frac{\varphi_1}{a} = y_1 + a_2\, y_1^2 + a_3\, y_1^3 + \ldots,$$

whereby

$$a_n = \frac{n\,(n - 3)\, a_{n-1} + 2\alpha_1^2\, a_{n-2} - \alpha_1^2\, a_{n-3}}{2\, n\,(n - 1)}$$

$$a_1 = 1, \quad a_2 = -\tfrac{1}{2}, \quad a_3 = \frac{\alpha_1^2}{6}, \quad a_4 = -\frac{\alpha_1^2}{18}$$

$$\varphi_2 = 1 + b_1\, y_1 + b_2\, y_1^2 + b_3\, y_1^3 + \ldots - \frac{\varphi_1}{a} \log y_1\,,$$

whereby

$$b_n = \frac{n(n-3)b_{n-1} + 2\alpha_1^2\, b_{n-2} - \alpha_1^2\, b_{n-3} + 2(2n-1)a_n - (2n-3)a_{n-1}}{2\, n\,(n - 1)}$$

$$b_0 = 1, \quad b_1 = 0, \quad b_2 = -1 + \frac{\alpha_1^2}{2}, \quad b_3 = \frac{1}{8} + \frac{\alpha_1^2}{18} \ldots$$

For $\alpha_1 = 0$,

$$\varphi_1 = \frac{U - c}{2\,a},$$

$$\varphi_2 = -\,2a\,(U - c)\int^y \frac{dy}{(U - c)^2}$$

expressions which also represent the solutions for any desired U when $\alpha = 0$.

In the general consideration of φ_1 and φ_2 we first find that φ_1 remains regular even at the singular point, while φ_2 has a logarithmically infinite derivation at that point. If we remain, at least in so far as φ_2 is concerned, at some distance from the singular point $U = c$, then φ_1 and φ_2 approximate the complete disturbance equation (1) for a very large αR and all the closer, the greater αR is.* The error in φ_1 and φ_2 is small and of the order $(\alpha R)^{-1}$. On account of the infinity of the derivatives of φ_2, this conclusion is obviously no longer justified in the immediate vicinity of the critical point. We must therefore discuss the general disturbance equation (1) in the vicinity of $U = c$. This point is physically characterized by the fact that, due to the equality of the main velocity and the phasic velocity, any fluid particle always remains in the same pressure field, if the slight transverse motion is disregarded.

*We would arrive at the frictionless solutions in a more formal way, by inquiring after the integrals of the general disturbance equation, which, aside from the critical point $U = c$, remain constant with their derivatives for any increasing values of αR.

III. Solutions of the Disturbance Equation in the Vicinity of $U = c$

For the investigation of the solutions in the vicinity of the critical point $U = c$, we take a small region about this point, the "transition strip," in which $U - c$ can be replaced with sufficient accuracy by $U'_o\, y$ and likewise U'' by U''_o. We now introduce $y = (\alpha R U'_o)^{-1/3}\eta = \epsilon\,\eta$, whereby η may have very great values, due to the smallness of ϵ, even at small values of y. From equation (1) we then obtain the differential equation in η:

$$i\,\varphi'''' + \varphi''(\eta - \epsilon^2 2\alpha^2 i) - \varphi\left(\epsilon\frac{U''_o}{U'_o} + \epsilon^2\eta\alpha^2 - \epsilon^4\alpha^4 i\right) = 0, \qquad (3)$$

so that we obtain, in close approximation, solutions of the disturbance equation for large αR (i.e., small ϵ) in the vicinity of $U = c$ from the differential equation

$$-\,i\,\varphi'''' = \eta\,\varphi'' - \epsilon\frac{U''_o}{U'_o}\varphi \qquad (4)$$

since, in the differential equation (3), $U = c$ (i.e., $\eta = 0$), there is no singular point, and approximate values of the coefficients of the differential equation must therefore yield approximate solutions. The differential equation (4) yields no correction for φ_1, but does for φ_2, which is obtained as follows.

At the critical point y (and also η) $= 0$, we put $\varphi_2 = 1$. This value, as an approximation of φ_{2c}, we introduce into the

term $\epsilon \frac{U''_0}{U'_0} \varphi$ and calculate from differential equation (4) a correction which we will designate by $\epsilon \varphi_{21}$. The differential equation for φ_{21} is then

$$- i \varphi''''_{21} = \eta \varphi''_{21} - \frac{U''_0}{U'_0} .$$

Since it may be assumed that the frictionless solution is applicable at some distance from the critical point, we will try to obtain a solution of equation (5) which is based outwardly on the corresponding frictionless solution. In the vicinity of the critical point (i.e., for small $|y|$) a frictionless calculation yields $\varphi''_{21} = \frac{U''_0}{U'_0} \frac{1}{\eta}$. This value must serve therefore as the basis for the solution of equation (5) outwardly; for example, for large $|\eta|$.

If, by way of abbreviation, we designate by $H^{(1)}_{1/3}$ and $H^{(2)}_{1/3}$ the Hankel functions of the first and second form with the subindex 1/3 and the argument $\frac{2}{3} (i\eta)^{3/2}$: $H^{(1)(2)}_{1/3} [- (i\eta)^{3/2}]$, then the solutions of the homogeneous differential equation (5), namely

$$\eta^{1/2} H^{(1)}_{1/3} \text{ and } \eta^{1/2} H^{(2)}_{1/3} \tag{6}$$

yield a solution of the inhomogeneous differential equation (5)

$$\frac{U''_0}{U'_0} \frac{\pi}{6} \eta^{1/2} \left\{ H^{(2)}_{1/3} \int^{\eta} \eta^{1/2} H^{(1)}_{1/3} \, d\eta - H^{(1)}_{1/3} \int^{\eta} \eta^{1/2} H^{(2)}_{1/3} d\eta \right\} \tag{7}$$

For the convenient representation of the Hankel functions, we will consider this solution in the lower η half-plane (which is permissible for the regularity of the solution), so that

$0 \geqq \arg \eta \geqq -\pi$. Moreover, $\frac{3\pi}{4} \geqq \arg [(i\eta)^{3/2}] \geqq -\frac{3\pi}{4}$. For the connection with the frictionless solution, we utilize the behavior of equation (7) for large $|\eta|$. At the junction points $\eta = \pm r$. On the circle with this radius, $\eta = r\, e^{i\vartheta}$. The lower limit of both integrals of equation (7) is located at a point of this semicircle in the lower half-plane. We replace the Hankel functions by their asymptotic formulas for large values of the argument

$$H_{1/3}^{(1),(2)} = \left(\frac{3}{\pi}\right)^{1/2} (i\eta)^{-3/4} \exp\left\{\pm i \left[\frac{2}{3} (i\eta)^{3/2} - \frac{5}{12}\pi\right]\right\}$$

Thereby $-\frac{3\pi}{8} \leqq \arg [(i\eta)^{-3/4}] \leqq \frac{3\pi}{8}$. The abbreviation "exp" denotes the exponential function of the bracketed expression. For the following, we need equation (7) on the semicircle in the lower η half-plane with r. The first integral in equation (7) there attains, in the first asymptotic approximation, the value

$$-\left(\frac{3}{\pi}\right)^{\frac{1}{2}} e^{-\frac{3i\pi}{8}}\, r^{-\frac{3}{4}}\, e^{-\frac{3i\vartheta}{4}} \exp\left\{+ i \left(\frac{2}{3} e^{\frac{3i\pi}{4}}\, r^{\frac{3}{2}}\, e^{\frac{3i\vartheta}{2}} - \frac{5\pi}{12}\right)\right\} + c_1(r)$$

The expression $c_1(r)$ and the subsequently used $c_2(r)$ are determined by the selection of the lower integral limit and are independent of φ. The second integral in equation (7) becomes

$$\left(\frac{3}{\pi}\right)^{\frac{1}{2}} e^{-\frac{3i\pi}{8}}\, r^{-\frac{3}{4}}\, e^{-\frac{3i\vartheta}{4}} \exp\left\{- i\left(\frac{2}{3}\, e^{\frac{3i\pi}{4}}\, r^{\frac{3}{2}}\, e^{\frac{3i\vartheta}{2}} - \frac{5\pi}{12}\right)\right\} + c_2(r)$$

If we now add to equation (7) solutions of the homogeneous differential equation, so that the terms with $c_1(r)$ and $c_2(r)$ drop out, we obtain φ''_{21}, because we then obtain along the

semicircle

$$\frac{U''_0}{U'_0}\frac{\pi}{6} r^{\frac{1}{2}} e^{\frac{i\vartheta}{2}} \frac{3}{\pi} r^{-\frac{3}{2}} e^{-\frac{3i\vartheta}{2}} 2 \exp\left\{- i \ldots\right\} . \exp\left\{+ i \ldots\right\}$$

$$= \frac{U''_0}{U'_0} r^{-1} e^{-i\vartheta} = \frac{U''_0}{U'_0}\frac{1}{\eta}$$

Thus we obtain the desired φ''_{21} from equation (7). We then obtain φ'_{21} by integration from φ''_{21} according to η, whereby, as the integration method, we again use the semicircle on which φ''_{21} is now known. This is

$$\varphi'_{21}\ (-r) = \varphi'_{21}\ (+r) - \frac{U''_0}{U'_0} i\ \pi .$$

In a similar manner it is finally found that the frictionless solution on the positive edge $(+r)$ of the transition strip becomes

$$1 + \frac{U''_0}{U'_0} y \log y$$

and on the negative edge $(-r)$

$$1 + \frac{U''_0}{U'_0} y \log |y| - \frac{U''_0}{U'_0} i\ \pi\ y$$

or, in general, that φ_2, known to be, for positive y, of the form

$$1 + \ldots + \frac{U''_0}{U'_0} \varphi_1 \log y,$$

in the transition to negative y, becomes

$$1 + \ldots + \frac{U''_0}{U'_0} \varphi_1 \log |y| - \frac{U''_0}{U'_0} i \pi \varphi_1$$

We have thus accomplished the transitional substitution in φ_2

at the critical point, which consisted in the introduction of an imaginary portion and therefore of a phasic jump in the x component of the disturbance. The above calculations show that the determination of the transitional substitution is equivalent to the determination of the physically real branch of the logarithm in φ_2.

The following statement may be made regarding the width of the transitional strip, for which we deduced the φ_{21} solution. r must be so large that the asymptotic approximations for large r, as used in the above calculation, are applicable up to an error which is admitted on account of the only approximate applicability of the frictionless solutions outside of the transitional strip.*

Calculation from the thus-determined r yields $2\epsilon r = 2(\alpha R U'_0)^{-1/3} r$, as the width of the transition strip in the original y coordinate. Hence, as soon as αR is large enough, the frictionless φ''_{21}, due to the narrowness of the transition strip which approaches 0 at the limit, actually acquires, on its boundaries, the values we have above put in the boundary conditions. Likewise φ_2, in the above integration method, sufficiently approximates the value 1 which we assumed for φ_{20}, thus verifying the asymptotic applicability of our formulas for large αR. Moreover, in Section VI, 3, reference is made to a somewhat different and expanded derivative of the transitional substitution.

*In the asymptotic development of φ''_{21} for large r_1, the term following $\frac{U''_0}{U'_0}\frac{1}{r_1}$ is $2\frac{U''_0}{U'_0}\frac{1}{r_1^4}$, from which the indicated error can be estimated.

Figure 1 gives for $U''_0/U'_0 = 1$ a graphically obtained qualitative picture of the behavior of φ'_{21} in the transition strip, divided into a real part $R(\varphi'_{21})$ and an imaginary part $I(\varphi'_{21})$.

The differential equations (4) and (5) give us directly the second pair of solutions, which follow from the general disturbance equation for large αR. Along with the solution φ''_{21} of equation (5), which we have just considered, there are obviously possible also the solutions (equation 6) of the corresponding homogeneous differential equation.

$$\varphi''_3 = \eta^{1/2} H^{(1)}_{1/3} \quad \text{and} \quad \varphi''_4 = \eta^{1/2} H^{(2)}_{1/3}$$

φ''_3 diminishes greatly for large positive values of η. It is written

$$\varphi_3 = \int_{+\infty}^{\eta} \int_{+\infty}^{\eta} \eta^{1/2} H^{(1)}_{1/3} \left[\frac{2}{3} (i\eta)^{3/2} \right] d\eta \, d\eta \qquad (8)$$

φ_4 approaches infinity for positive η and is not used in what follows. The error in φ_3 is of the order of magnitude of $\epsilon \dfrac{U''_0}{U'_0}$. According to their rate of variation φ_1 and φ_2 can also be designated as "slow" and φ_3 and φ_4 as "rapid."

IV. Solutions of the Disturbance Equation on the Wall for $c \gg U$

It will help our understanding of the nature of the four integrals, if we derive them in a special case independently of Section III. We will do this for the vicinity of the wall ($U = 0$) in the case of $c \gg U$, that is, when the critical point is outside of the layer under consideration. Hence, for a boundary layer in which c approaches $U - c$, U'' can likewise be replaced by U''_w, the value at the wall

$$- i \alpha R \left\{ c \left(\varphi'' - \alpha^2 \varphi\right) + U''_w \varphi \right\} = \varphi'''' - 2\alpha^2 \varphi'' + \alpha^4 \varphi \tag{9}$$

If the distance from the wall is designated by y_w, the solutions of this differential equation have the form $\varphi = e^{k y_w}$. We thus obtain both pairs of solutions in an especially clear manner. We have, namely,

$$k^2 = \frac{1}{2} \left\{ - i \alpha R c + 2\alpha^2 \pm \sqrt{- \alpha^2 R^2 c^2 - 4 U''_w i \alpha R} \right\}$$

$$= \frac{1}{2} \left\{ - i \alpha R c + 2\alpha^2 \pm \left[i \alpha R c - \frac{2U''_w}{c} + \frac{2i\, U''^{\,2}_w}{\alpha R c^3} \dots \right] \right\}$$

One k^2 accordingly approximates $\alpha^2 - \frac{U''_w}{c} + i \frac{U''^{\,2}_w}{\alpha R c^3}$ and gives us, as a result of the exclusive consideration of the largest terms with large αR, the pair of frictionless solutions, i.e., due to the dependence of the coefficients of the differential equation of y_w, only the beginning of their development from the wall out, approximately

$$1 + \left(\alpha^2 - \frac{U''_w}{c} \frac{y_w^2}{2!}\right) \text{ or } y_w + \left(\alpha^2 - \frac{U''_w}{c}\right) \frac{y_w^3}{3!}$$

The other k^2 becomes $-i\alpha R c$ and gives us φ_3 and φ_4. The solution corresponding to φ_3 in Section III becomes, up to a certain constant factor

$$e^{\frac{3i\pi}{4}\sqrt{\alpha R c y_w}} = e^{\frac{1}{\sqrt{2}}(-1+i)\sqrt{\alpha R c y_w}} \tag{10}$$

i.e., it diminishes very rapidly, so that the variation of the coefficients of the differential equation does not enter into consideration. It should be here noted that equation (10) can also be derived from the general equation (8) for φ_3 by specialization for negatively large η.

Tietjens derived φ_3 from a separate boundary-layer equation, while we simply derived all solutions from the general disturbance equation. φ_3 can be conceived as a sort of boundary-layer solution on the wall or, in other words, of outer frictional-layer solution. In the case $U'' \neq 0$, however, at the critical point where, according to the frictionless calculation, one of the disturbance components would become infinite, a second inner frictional-layer solution occurred which was derived as φ_{21} in Section III. In the above-mentioned special case of greater c, the two frictional layers also separate in space, but otherwise fit into each other. The essential difference in our investigation, as compared with that of Tietjens, resides in the appearance of the inner frictional layer.

V. Formulation of the Boundary Conditions

Having obtained an insight into the character of the four integrals of the disturbance equation, we will now revert to the boundary-value problem and formulate the boundary conditions on this basis.

We first take a profile $U(y)$, which increases from the value $U = 0$ on the wall to a maximum value, which then remains constant for any desired time. This, for example, holds very true for the important profile of the flow along a plate. Since φ_4 for positive y increases beyond all limits, it does not appear in the solution. For U = constant, the frictionless solutions are $e^{-\alpha y}$ and $e^{+\alpha y}$, that is, we can use only $e^{-\alpha y}$ due to the necessary limitation.* Hence $\varphi'/\varphi = -\alpha$ in this field, whereby we have accordingly obtained a simple boundary condition for the initial point of the field with constant U, which will be indicated in future by the subindex m. Since φ_3 has already disappeared, the condition is expressed by the equation

$$C_1 \ (\varphi'_{1m} + \alpha \ \varphi_{1m}) + C_2 \ (\varphi'_{2m} + \alpha \ \varphi_{2m}) = 0 \qquad (11)$$

or, in an abbreviated form, by

$$C_1 \ \phi_{1m} + C_2 \ \phi_2 = 0 \qquad (11a)$$

φ and φ' must now vanish at the wall. If we designate the values there by the subindex w and simultaneously apply

*$e^{+\alpha y}$ and φ_4 cannot balance each other, because $\alpha < \sqrt{\alpha R}$ according to a remark in Section I.

the transitional substitution in φ_{2w} and in φ'_{2w}, we obtain

$$C_1 \ \varphi_{1w} + C_1 \ \varphi_{2w} + C_3 \ \varphi_{3w} = 0 \tag{12a}$$

and

$$C_1 \ \varphi'_{1w} + C_2 \ \varphi'_{2w} + C_3 \ \varphi'_{3w} = 0 \tag{12b}$$

or

$$C_1 \ \varphi_{1w} + C_2 \ \varphi_{2w} - \frac{\varphi_{3w}}{\varphi'_{3w}} (C_1 \ \varphi'_{1w} + C_2 \ \varphi'_{2w}) = 0 \tag{12c}$$

The ratio $\frac{\varphi_{3w}}{\varphi'_{3w}}$ has already been calculated in the work of Tietjens and can be expressed by the symbol D. It then becomes

$$\frac{\varphi_{3w}}{\varphi'_{3w}} = - \epsilon \ D \tag{13}$$

D is here introduced as a function of z_0, which, in the η system of coordinates (Section III), is the measured distance of the wall from the critical point, which can be designated by η_w, hence

$$\eta_w = - \frac{c}{U'_0} \frac{1}{\epsilon} \tag{14}$$

The following table gives the values according to Tietjens:

η_w	$D(\eta_w)$	η_w	$D(\eta_w)$
0	0.702 - 0.425 i	-3.0	1.400 + 0.515 i
-0.5	0.785 - 0.411 i	-3.5	1.180 + 1.130 i
-1.0	0.920 - 0.389 i	-4.0	0.460 + 1.250 i
-1.5	1.043 - 0.297 i	-4.5	-0.0405 + 0.8080 i
-2.0	1.206 - 0.147 i	-5.0	0.0057 + 0.3645 i
-2.5	1.357 + 0.108 i	-5.5	0.1913 + 0.2393 i

The two conditions (11a) and (12c) yield a determinant relation which can be calculated from φ_{3w}/φ'_3. We obtain thereby

$$- \epsilon\ D = \frac{\Phi_{1m}\ \varphi_{2w} - \Phi_{2m}\ \varphi_{1w}}{\Phi_{1m}\ \varphi'_{2w} - \Phi_{2m}\ \varphi'_{1w}}$$

and, on dividing by η_w,

$$- \frac{D}{\eta_w} = - \frac{U'_0}{c} \frac{\Phi_{1m}\ \varphi_{2w} - \Phi_{2m}\ \varphi_{1w}}{\Phi_{1m}\ \varphi'_{2w} - \Phi_{2m}\ \varphi'_{1w}} \qquad (15)$$

The right side of equation (15) is known, when we write the parameters c and α before it. It is quite generally designated by $E(c:\alpha)$. The left side is expressed as a function of η_w. We now construct a polar curve from $- \frac{D(\eta_w)}{\eta_w}$ with its real part as abscissa and its imaginary part as ordinate (Fig. 3). We then select a definite c and plot E as a function of α in the polar curve, whereupon we determine the intersection points with the $- \frac{D\ (\eta_w)}{\eta_w}$ polar diagram and the η_w corresponding to the "transmission point" (Section I). From η_w we can then determine ϵ according to equation (14) and consequently the last parameter R.

Regarding D it should be noted that, for a negatively large η, it is converted into $\frac{e^{i\pi/4}}{\sqrt{-\eta_w}}$. This can be accomplished either through an asymptotic consideration of the expression for φ_3 as given in equation (8) or from the directly derived equation (10) in Section IV, if c is there finally expressed by η_w.

For a symmetrical U profile we obtain a simple form of the boundary condition from the fact that in this case φ can be resolved into an even and an uneven part, each of which must

of itself satisfy the disturbance equation and the boundary conditions. Hence, we have in the middle, since φ_4 cannot be introduced here and φ_3 has already vanished, the condition

$$C_1 \varphi'_{1m} + C_2 \varphi'_{2m} = 0 \tag{16}$$

for the even φ, or

$$C_1 \varphi_{1m} + C_2 \varphi_{2m} = 0 \tag{17}$$

for the uneven φ, which takes the place of equation (11), while the condition at the wall (equation 12) is maintained.

Tietjens expressed the one boundary condition on the wall, and indeed for the y component according to a suggestion by Prandtl in a different form from ours. Since doubt has occasionally been expressed regarding the correctness of this boundary condition, we hereby confirm its agreement with our own conclusions.[1)] We will describe the Tietjens method formally and refer to the work itself for its physical justification. Tietjens does not let his frictional-layer solution for $\eta = +\infty$, which would correspond to our φ_3 and be designated by φ_3^* (proportional to Tietjens' v^*), go toward 0 like ours, but toward a constant value $\varphi_{3\infty}^*$ (proportional to v_∞^*), so that $\varphi_3^* = \varphi_3 + \varphi_{3\infty}^*$; for $\varphi_3^{*\prime}{}_\infty$, like φ'_3, is also put equal to 0. Then the constant $\varphi_{3\infty}^*$ is so determined by Tietjens that $\varphi_{3w}^* = 0$, that is, $\varphi_{3\infty}^* = -\varphi_{3w}$. Lastly, the y component, which follows from the frictionless solutions for the wall, is

1) From here on, the footnotes are designated by numbers.

Noether, Z. f. angew. Math. u. Mech., Vol. VI, 1926, p.243, and the accompanying discussion between Prandtl and Noether, ibidem, pp.339-340.

put equal to v_∞^* that is,

$$C_1 \varphi_{1W} + C_2 \varphi_{2W} = C_3 \varphi_{3\infty}^*$$

while we above put

$$C_1 \varphi_{1W} + C_2 \varphi_{2W} + C_3 \varphi_{3W} = 0$$

Both conditions are therefore identical according to the relation between $\varphi_{3\infty}^*$ and φ_{3W}.

VI. Equilibrium of Flow along a Flat Plate

1. The above method of calculating will be applied first to the velocity profile which develops at large Reynolds Numbers on a plate immerged parallel to a flow of infinite extent, the leading edge of the plate being perpendicular to the direction of flow. We choose this particular velocity distribution, because experiments on its stability are in progress and because the assumptions for the accuracy of our approximation are here especially favorable. This profile was calculated by Blasius according to Prandtl's theorems[2] and is characterized especially by the fact that it begins at the wall with a vanishing curvature and joins the undisturbed velocity outwardly with a sharp asymptote corresponding to the asymptotic behavior of the error integral (Fig. 2).

For our purposes it will be useful to have an approximate representation of the profile. According to the above, the sim-

[2] L. Prandtl, "Ueber Flüssigkeitsbewegung bei sehr kleiner Reibung," Verh. d. III Internat. Math. Kongresses, Heidelberg, 1904; reprinted in Vier Abhandlungen zur Hydrodynamik und Aerodynamik, Göttingen, 1927. For translation, see N.A.C.A. Technical Memorandum No. 452.

plest way is to let the velocity distribution begin at the wall with a straight section and to replace the asymptotic junction with the undisturbed velocity by a finite junction (at M in Fig. 2), i.e., to continue the straight section by a parabolic section. If we designate by y_m the distance from the vertex of the parabola, we then allow, toward the one side, U constant to continue with its maximum value (1). Toward the other side $U = 1 - y_m^2$, if we correspondingly select the reference quantities with which we form the absolute quantities used. Toward the wall and indeed from $y_m = 0.84$ (at A in Fig. 2), we then have $U = 1.705 - 1.680\ y_m$, so that $y_m = 1.015$ at the wall $(U = 0)$. Thereby U itself is represented with sufficient accuracy, but we cannot put $U' =$ constant and $U'' = 0$ in the region near the wall. With y_m as the wall distance corresponding to the first terms of the Blasius series, the values

$$U' = 1.68\ (1 - 3.65\ y_w^3),$$

$$U'' = -\ 18.4\ y_w^2,$$

apply, instead, near the wall, in so far as these quantities are subsequently used.

The frictionless solutions φ_1 and φ_2 are calculated according to the series formulas of Section II for the parabolic distribution of U and with the utilization of the solutions $\sin \alpha y$ and $\cos \alpha y$ for linear U. It was thus possible, aside from the transitional substitution, to present the solu-

tions for the wall and the parabola vertex with sufficient accuracy[3]. The transitional substitution cannot, of course, be calculated with the roughly approximated profile (straight line + parabola), but with the more accurate value of $\frac{U''_0}{U'_0} = -3.9\, c^2$, which is obtained from the indicated formulas for U'' and U', when, according to the above, $c = U = 1.68\, y_w$ even at the critical point.

2. The intersection points of the E curves (Section V) with the $-\frac{D}{\eta_w}$ polar diagram are next determined for $c \sim 0$. An imaginary component is introduced into E by the phasic shift accompanying $\frac{U''_0}{U'_0} \sim c^2$, so that E is purely real for sufficiently small c. The intersection points with the $-\frac{D}{\eta_w}$ diagram are therefore either at $-\frac{D}{\eta_w} = 0.56$, or $\eta_w = -2.3$, or $-\frac{D}{\eta_w} = 0$ and negatively very large η_w. By considering only the linear terms in c and α, we obtain

$$E = \frac{-1.19\, \alpha + 2\, c}{0.596\, c\, 3.36}$$

For the first intersection point with $-\frac{D}{\eta_w} = 0.56$ we accordingly have $\alpha = 0.74\, c$ and, under consideration of $\eta_w = -2.3$ or

$$(1.68\, \alpha\, R)^{1/3}\, 0.596\, c = 2.3$$

$$R = 46\, c^{-4}$$

According to equation (18) we have $\alpha = 1.68\, c$ for the second

[3]A special expression is required by this method for the representation of φ_2, when the critical point $(U=c)$ approaches very near to the junction point of the straight line and parabola. Nevertheless, we can here dispense with the repetition of this consideration, which is important for the numerical calculation but without interest for the problem as a whole.

intersection with $\frac{D}{\eta_w} = 0$. In this region η_w is very large and therefore $\epsilon D = (2 \alpha R c)^{-\frac{1}{2}} (1 + i)$ according to equation (10). For the determination of R we must here include the higher terms in c and α and utilize therefore the imaginary part of D, to which the imaginary part of $- E \eta_w$ should be equal for the transmission points. This becomes $4.36 \, i \, c^4$ on the basis of the just-calculated α, so that $R = 0.0156 \, c^{-10}$ We can now calculate η_w and obtain $0.21c^{-2}$, that is, for small c actually as large values as warranted by the utilization of asymptotic value of D.

Disregarding the explicit representation of the relation between R, α and c, both these limiting cases are important, because our asymptotic calculations for large αR apply to them with great accuracy. The formulas, moreover, yield a utilizable approximation up to about $c = 0.05$.

The other transmission points are found by the graphic method developed in Section V (Fig. 3). Beyond a certain c value (e.g., in Fig. 3 at $c = 0.43$), the E curves yield no further intersection point with the $- \frac{D}{\eta_w}$ diagram, though they otherwise yield two intersection points. Before entering into the physical discussion, we will add a few more remarks regarding the accuracy of our calculation.

3. Our whole calculation is only an approximation, which is poorer in proportion to the smallness of the αR values. We have continually worked with the transitional substitution

of Section III, although the transitional strip does not by any means always remain small, so that we can no longer assume $\frac{U''}{U'}\varphi_2$ to be approximately constant in the transitional strip, and even the junction conditions have to be changed as regards the greater $|y|$ values on the edge of the transitional strip. Now, however, the transitional substitution can be accomplished under somewhat different conditions. As the differential equation for φ_2 in η in the transitional strip, we can put

$$i\varphi'''' + \eta\varphi''_2 = \varphi_2\left[\epsilon\frac{U''}{U'_0} + \epsilon^2\alpha^2\eta\left(1 + \frac{U''_0}{U'_0}\frac{\epsilon\eta}{2!} + \frac{U'''_0}{U'_0}\frac{\epsilon^2\eta^2}{3!} + \dots\right)\right]$$
$$-\varphi''_2\,\eta\left(\frac{U''_0}{U'_0}\frac{\epsilon\eta}{2!} + \frac{U'''_0}{U'_0}\frac{\epsilon^2\eta^2}{3!} + \dots\right) \tag{19}$$

in which the U derivatives are the original ones according to y.[4] The right-hand member of equation (19) is no longer constant in the transitional strip. We can therefore show that there is a solution of equation (19) which, for large $|\eta|$, joins the frictionless solution φ_2 with the branch of the logarithm previously known as physically real. By considering the right-hand member of equation (19) as an inhomogeneous term of a differential equation of the second order for φ''_2, it is shown, with the same means (asymptotic discussion of Hankel functions and of the corresponding integrals) as in Section III, that, on the semicircle there introduced, the equation

$$\varphi''_2 = \varphi_2\left[\epsilon\frac{U''}{U'_0\eta} + \epsilon^2\alpha^2\left(1 + \frac{U''_0}{U'_0}\frac{\epsilon\eta}{2!} + \dots\right)\right] - \varphi''_2\left(\frac{U''_0}{U'_0}\frac{\epsilon\eta}{2!} + \dots\right)$$

4) See footnote, page 26.

is identical with the frictionless-disturbance equation, whereby the transitional substitution is confirmed, even under these general assumptions. We commit a slight error, however, by taking φ_2 at the wall as already frictionless (naturally, with inclusion of the phasic jump), i.e., by assuming that the frictional effect has vanished. Outside of the transitional strip (for example, at the point with the index m) the errors in φ_1 and φ_2, through disregard of friction of the order $(\alpha R)^{-1}$, are always very small. On the contrary, the error in φ_3, of the order $\epsilon \frac{U''_0}{U'_0}$, is 0.055 in the most unfavorable case.

This rough estimation of the errors in our approximation shows that our calculation yields the physically correct fundamental values, while indicating at the same time that, for the last intersection points (small Reynolds Numbers), the numerical values of the disturbance parameters and of ϵ are subject to errors of a few per cent, especially in the determination of R from ϵ.

It may seem surprising that, in our calculation, the velocity profile on the plate was assumed to depend only on y, although there is really a slight dependence on x. It can be shown, however, that the consideration of this dependence, for the flow along the plate, only introduces terms of the order of magnitude of $(\alpha R)^{-1}\varphi'''$, $(\alpha R)^{-1}\varphi'$, $(\alpha R)^{-1}\alpha^2\varphi'$ and still

4) Footnote from page 25.
The two frictional terms $\epsilon^2 2\alpha^2 i\varphi''_2$ and $-\epsilon^4\alpha^4 i\varphi_2$, omitted on the right side of equation (19), can also be easily included in the following consideration, though in our case they do not appreciably affect the transitional substitution.

smaller ones, which would have very little effect on the results of our calculation.

4. In giving the results of our calculation of the transmission points, we will write in full the hitherto abbreviated absolute quantities. Instead of the previously used, somewhat arbitrarily determined reference length (half the width of a certain parabola), we will introduce a physically logical reference length. The determination of a characteristic length from the present and other boundary-layer profiles is rendered difficult by the asymptotic union with the undisturbed velocity, so that nearly all of the previous definitions of the boundary-layer thickness are not rational. The only formula for the boundary-layer thickness, which is not arbitrary, which can be accurately determined by experiment, and which covers the entire course of the profile is

$$\delta = \int \left(1 - \frac{U}{U_m}\right) dy,$$

in which the integral extends from the wall to the undisturbed potential velocity U_m. This length, formerly termed also displacement thickness but here simply boundary-layer thickness, yields, when multiplied by U_m, just the measure for the reduction of the flow by the boundary-layer friction. In this case δ is 0.341 of our former reference length. We accordingly take $\frac{U_m \delta}{\nu}$ as Reynolds Number R.

According to our determination of the transmission points,

two disturbance parameters correspond to each R, except that only one disturbance parameter corresponds to the last critical Reynolds Number 420. Figures 4-5 show the disturbance parameters $\alpha\,\delta$, $\frac{c}{U_m}$, $\frac{\beta\delta}{U_m}$ plotted against the logarithmic R. The space enclosed in the curves is obviously the region of the unstable oscillations. The subindex r of β and c indicates that the diagrams cover only the real part of these quantities.

Here a few transmission points are indicated numerically as obtained from our calculation in connection with Figure 3.

$\frac{c}{U_m}$	$\alpha\delta$	$\frac{\beta\delta}{U_m}$	R
0.4	0.272	0.109	445
0.4	0.368	0.147	716
0.2	0.076	0.015	7330
0.2	0.160	0.032	34800

Moreover, we will repeat the formulas for very large R, although they hardly continue to be physically important.

$$\frac{c}{U_m} = \left(\frac{R}{15.7}\right)^{-1/4}, \quad \alpha\delta = 0.252\,\frac{c}{U_m}$$

and

$$\frac{c}{U_m} = \left(\frac{R}{0.0053}\right)^{-1/10}, \quad \alpha\delta = 0.573\,\frac{c}{U_m}$$

for the other branch of the equilibrium curve.

We first establish the fact that an extremely narrow oscillation field is very dangerous for the laminar flow. Just as there is a lower limit of 420 for R, there is an upper limit for the disturbance parameter, beyond which there is no further

instability: $\frac{c}{U_m} = 0.425$, $\alpha\delta = 0.367$, $\frac{\beta\delta}{U_m} = 0.148$. The great wave lengths λ of the disturbance are especially striking in comparison with the boundary-layer thickness. $\lambda = \frac{2\pi}{\alpha}$, so that minimum $\lambda = \frac{2\pi}{0.367}\delta = 17.1\delta$. Also certain observations of Prandtl regarding the previously mentioned channel, at the beginning of which the flow somewhat resembles the plate flow, seem to refer to these great wave lengths. Since δ increases along the plate, it is perhaps desirable to assign to λ a limit, which is independent of x. According to Figure 4, a certain region of unstable $\alpha\delta$ corresponds to every $\frac{U_m\delta}{\nu}$. We can eliminate δ from $\alpha\delta$ and $\frac{U_m\delta}{\nu}$ and then determine the maximum unstable α or the minimum λ for any given U_m and ν. The minimum unstable λ is $8400\frac{\nu}{U_m}$, which is approximately proportional to the steady oscillation at $R = 420$. If the length of the plate from its leading edge to the point $R = \frac{U_m\delta}{\nu} = 420$ is designated by l_1, then $l_1 = 59000\frac{\nu}{U_m}$, because $\delta = 1.73\, l\sqrt{\frac{\nu}{U_m l}}$. Then $\lambda = l_1/7$.

Tietjens investigated the stability of a profile which ascends linearly to U_m and then continues with a bend. This profile must therefore be regarded as a rough approximation of ours. The transmission points were taken from his Figure 17 and plotted, in part, in our Figure 4 with crosses, whereby, according to Tietjens, $b_1 = 2\delta$. Tietjens obtained one branch of the equilibrium curve in very close approximation. Since he obtained only this one branch, he found no limit for R nor for

$\alpha\delta$ which depends on the general disappearance of the bend in the Tietjens profile.

Heisenberg[5] undertook to investigate the stability of profiles with bends. In general, Heisenberg assumes that $\alpha = 0$. The convergence of the development of the solutions according to α is not explained. The transitional substitution for φ_2 (Heisenberg φ_4) is accomplished by another and not quite convincing method. The numerical value of the phasic bend agrees with ours in his special case. Heisenberg did not calculate a profile, but confined himself to suppositions which, in part (as regards the existence of an upper limit for α and a lower limit for R) lie in the same direction as our results concerning the special profile for the flow along the plate.

5. The comparison of our results with those of Burgers[6] and Zijnen[7] and Hansen[8] is incomplete for two reasons. In the first place we are not well enough informed concerning the actual disturbances, and then it is not clear as to how far the turning point defined by these writers agrees with our beginning of instability of the laminar flow. Both Burgers and Hansen define the turning point as follows. On the forward portion of the plate there is always a laminar section, as determined by Prandtl and Blasius. After the turning point, there is a decided

5) "Ueber Stabilität und Turbulenz von Flüssigkeitsstromen," Ann. d. Phys. IV, Vol. 74 (1924), p.597.

6) J. M. Burgers, Proc. of the First Intern. Congress for Applied Mechanics, Delft, 1924, p.113.

7) B. G. van der Hegge Zijnen, Thesis, Delft, 1924.

8) M. Hansen, Z. f. angew. Math. u. Mech., Vol. 8 (1928), p.185.

turbulent portion, whose laws were explained by Prandtl and Karman (1/7 power law, etc.). In both sections the velocity gradient close to the wall continually diminishes downstream with an increase only in the transition zone. The critical Reynolds Number is calculated simply from the minimum point of the velocity gradient, while the instability of the laminar flow might begin sooner at smaller δ and hence at smaller R. At best we may therefore expect only a confirmation of our calculated critical Reynolds Numbers. It should be noted that the boundary-layer thickness obtained by these investigators is 3.18 times that obtained by us. On the average, they obtained in our scale a critical Reynolds Number of about 950[9] (as calculated from their value of 3000), which decreased with greater disturbance to 500 (near a disturbing screen placed in the air stream at the leading edge of the plate). If it is considered that, in our calculation at 420, there was still a single partial undamped oscillation, while, in the case of the experimentally determined turning point, there was already a considerable increase in the small disturbances, it can then be said, under consideration of the above critical remarks, that our calculated results agree very well with the experimental results.

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9) Since our $R = \frac{U_m \delta}{\nu} = 1.73 \sqrt{\frac{U_m l}{\nu}} = 1.73 \sqrt{R_l}$ where l represents the plate length from the leading edge, this numerical value corresponds to an R_l of 300,000.

For comparing with the experiments it is important to know the degree of acceleration or damping and also the distribution of the oscillation amplitude. This and other questions, especially the behavior of other profiles, will be discussed in a second report.

Translation by Dwight M. Miner,
National Advisory Committee
for Aeronautics.

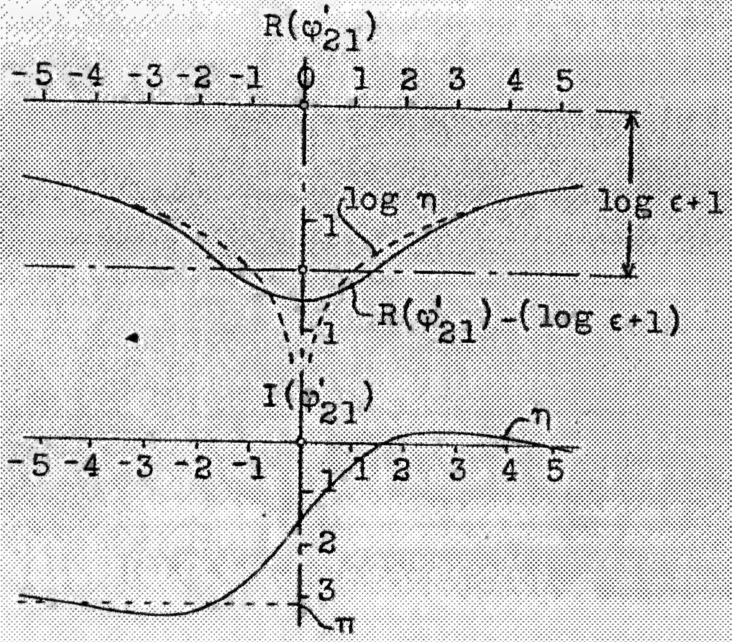
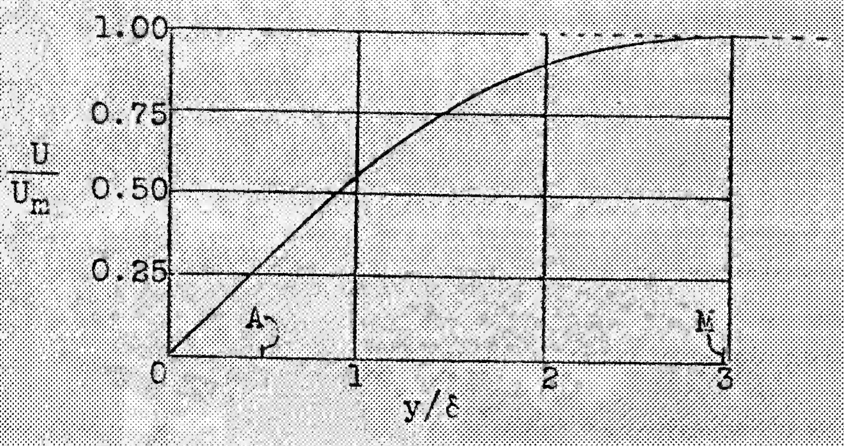

R(φ'21)
-5 -4 -3 -2 -1 0 1 2 3 4 5
log η
log ϵ+1
1
R(φ'21)-(log ϵ+1)
1
I(φ'21)
η
-5 -4 -3 -2 -1 1 2 3 4 5
1
2
3
π

Fig.1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0.75
0.50
0.25
U/Um
A
M
0
1
2
3
y/δ

Fig.2

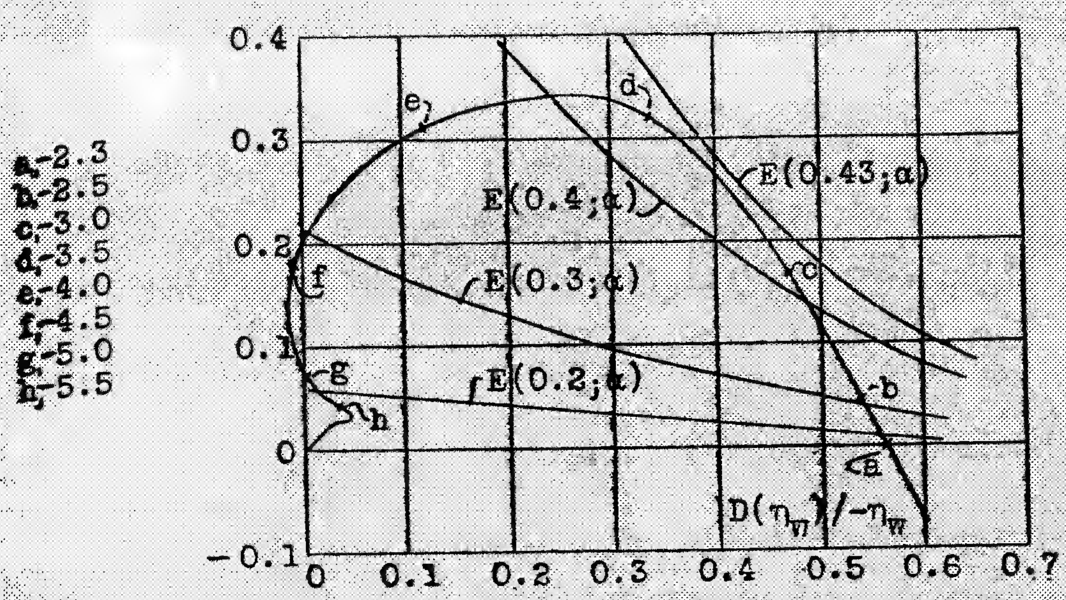

Fig.3

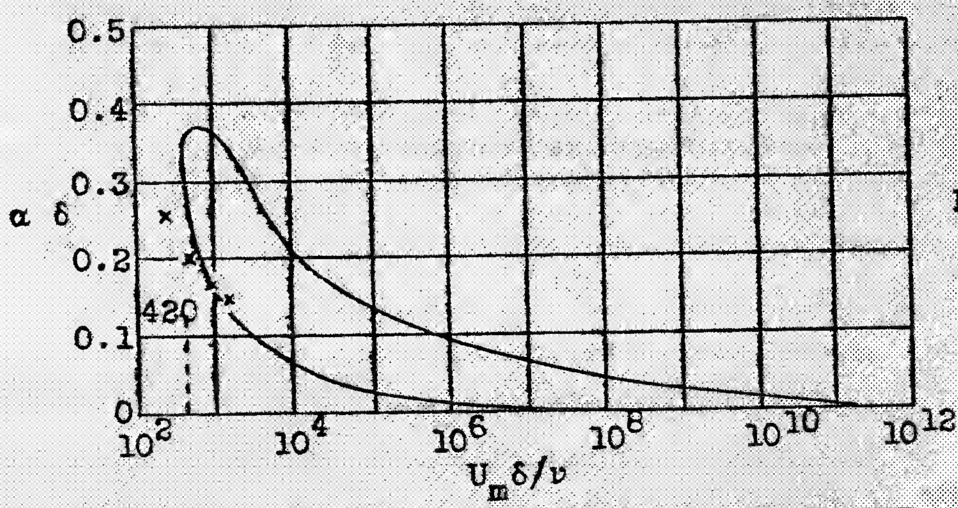

Fig.4

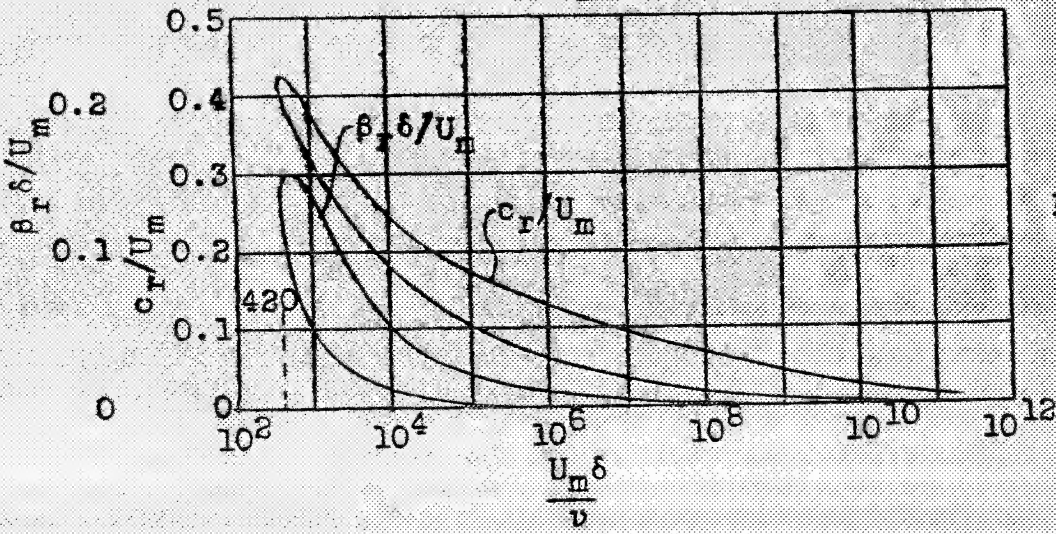

Fig.5